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A deformed oscillator with Coulomb energy spectrum

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Abstract. A deformed oscillator, with eigenvalues equal to the eigenvalues of the Schrödinger equation with the Coulomb potential, is constructed. The deformed oscillator algebra has a polynomial realization and the associate deformed operations of integration and differentiation are studied.

1. Introduction

The q-deformed algebras $SU_q(2)$, $U_q(1, 1), \ldots$ were introduced by Kulish and Sklyanin (1982), Kulish and Reshetikhin (1983), Sklyanin (1982) and Jimbo (1985, 1986) as a mathematical tool useful for the solution of the Yang–Baxter equation. A collection of the original papers on this subject are assembled in Jimbo (1990). Biedenharn (1989) introduced the q-deformed harmonic oscillator and constructed a realization of the $SU_q(2)$; this was done independently by Macfarlane (1989).

Initially, the q-deformed harmonic oscillator was considered as an intermediate step to study the deformed quantum algebras as $SU_q(2)$, $U_q(1, 1), \ldots$ whose applications are relevant in inverse problems and in other branches of physics. The quantum mechanical systems whose properties could be described by the q-deformed oscillator, are the subject of recent investigations. Floratos and Tomaras (1990) showed that if a particle moves in the field of a shielded magnetic flux on a discretized cycle, then its Hamiltonian corresponds to the energy spectrum of a q-deformed oscillator. Bonatsos *et al* (1991b) have studied the hydrogen molecular spectrum, using the q-deformed anharmonic oscillator energy spectrum. Bonatsos *et al* (1991a) found that the potential with the same WKB spectrum as the q-deformed oscillator has similarities with the modified Pöschl–Teller potential. It is worth noticing that the Pöschl–Teller potential has the same energy spectrum as the fermionic oscillator, studied by Ohnuuki and Kamefuchi (1982). The q-deformed oscillator constitutes a special type of deformation of the ordinary harmonic oscillator, but there are other deformation schemes known in the literature, such as the deformed oscillator given by Arik and Coon (1976), the oscillator given by Ohnuuki and Kamefuchi (1982), its q-deformed version given by Floreanini and Vinet (1990a), and the two-parameter deformed oscillator studied by Chahrabati and Jagannathan (1991) and by Jannousis *et al* (1991). These oscillator algebras can be studied by a unified generalized oscillator scheme given by several authors (Odaka *et al* 1991, Jannousis 1990, Beckers and Debergh 1991, and Daskaloyannis 1991). The construction of generalized deformed oscillators corresponding to well known potentials and the study of the correspondence between the properties of the conventional potential picture and the algebraic one, using deformed oscillators, seems to be an attractive field of investigation. The correspondence between the algebraic and the potential picture for an energy spectrum is analogous to the correspondence of the potential model and the creation and destruction operator algebra in the harmonic oscillator case. The construction of a deformed oscillator corresponding to the Pöschl–Teller energy spectrum is studied by Daskaloyannis (1992). The idea of using algebraic methods for solving quantum mechanical problems (exactly or partially) was studied by different authors, a reference review is given by Shifman (1989), who proposed the term *algebraization of the spectral problem*, and related references are cited there. In these investigations, the quantum Hamiltonians are written using quadratic and linear combinations of the generators of known classical algebras as SU(2), O(4), etc. A relevant paper treating the bound and scattering states of the Pöschl–Teller potential is given by Alhassid *et al* (1983). The difference between the method which is proposed in this paper and the above-cited references arises from the fact that here we propose an *ab initio* analytical construction of an oscillator algebra and of the corresponding Fock space; the constructed deformed oscillator has the same energy spectrum for a given potential.

The energy spectra can be classified in three categories:

(i) The number of the energy eigenvalues is infinite and the energy spectrum does not have any finite accumulation point. This is the case of the harmonic oscillator or of the q-deformed oscillator, with q being a real positive number. The algebra of the destruction, creation and number operators of these cases is very well known and the Fock space is well defined. The differentiation and the integration on the polynomial representation on the harmonic oscillator define an analysis which is the usual analysis (i.e. Taylor theorem, special functions theory, etc.), while in the case of the q-deformed oscillator the corresponding analysis is the same as the q-analysis, worked by the mathematicians many years ago (Exton 1983, Andrews 1986). For more recent literature on this subject see Floreanini and Vinet (1990b, 1991) and Bracken *et al* (1991).

(ii) The energy spectrum is a finite set. This is the case of the q-deformed oscillator, with q being a root of unity (see Yan Hong 1990, Jian-Hui *et al* 1991, Baulieu and Floratos 1991) or the case of the Pöschl–Teller or Morse potential (see Daskaloyannis 1992).

(iii) The energy spectrum is a set with a denombrable set of points with one accumulation point, which can be shifted to zero. This is the case of the Coulomb potential. This special category is the subject of this paper.

The paper is organized as follows: In section 2, we present a short description of the generalized deformed oscillator algebra. In section 3 the deformed oscillator algebra, having an energy spectrum equivalent to the Coulomb energy spectrum, is constructed. In section 4 the associated deformed differentiation and integration are studied.

2. The generalized deformed oscillator algebra

A general deformation of the harmonic oscillator can be given by the basic relation

$$f(aa^\dagger)-f(a^\dagger a)=1 \tag{1}$$

where a and $a^\dagger$ are conjugate operators, $f(x)$ is a real analytic function defined on the real positive axis. In the ordinary oscillator algebra the function $f(x)$ is defined by

$$f(x)=x \tag{2}$$

which leads to the commutation relation

$$[a, a^\dagger] = 1.$$

The *number* operator N, by definition, satisfies the commutation relations

$$[a, N] = a \qquad \text{and} \qquad [a^\dagger, N] = -a^\dagger. \tag{3}$$

It can be shown that this operator is given by the relation

$$N = f(a^\dagger a). \tag{4}$$

If equation (1) is true then the following relation is also true:

$$aa^\dagger = g(a^\dagger a) \tag{5}$$

where the function $g(x)$ is defined by

$$g(x) = F(1 + f(x)) \qquad \text{and} \qquad F = f^{-1}. \tag{6}$$

By induction the following relations can be proved:

$$[a, (a^\dagger a)^n] = ((g(a^\dagger a))^n - (a^\dagger a)^n)a$$

and

$$[a^\dagger, (a^\dagger a)^n] = -a^\dagger((g(a^\dagger a))^n - (a^\dagger a)^n).$$

These equations imply

$$[a, f(a^\dagger a)] = (f(g(a^\dagger a)) - f(a^\dagger a))a \tag{7a}$$

and

$$[a^\dagger, f(a^\dagger a)] = -a^\dagger(f(g(a^\dagger a)) - f(a^\dagger a)). \tag{7b}$$

Thus the *number* operator $N = f(a^\dagger a)$ satisfies equations (3).

Assume that $|\alpha\rangle$ is a base of eigenvectors of the number operator N

$$N|\alpha\rangle = \alpha|\alpha\rangle. \tag{8}$$

Then equations (3) imply that the operator a (or $a^\dagger$) is a destruction (or a creation) operator such that

$$a|\alpha\rangle = \sqrt{[\alpha]}\,|\alpha - 1\rangle \qquad a^\dagger|\alpha\rangle = \sqrt{[\alpha + 1]}\,|\alpha + 1\rangle \tag{9}$$

where $[\alpha]$ is a function of α; furthermore from equation (6) we can find

$$[\alpha + 1] = g([\alpha]) \qquad \text{or} \qquad f([\alpha + 1]) = 1 + f([\alpha]). \tag{10}$$

Thus finally from these equations, we conclude

$$[\alpha] = F(\alpha). \tag{11}$$

The function $F(x)$ is characteristic of the deformed oscillator. The knowledge of this function determines all the properties of the oscillator, just as in the case of Lie algebras the knowledge of the structure constants determines the properties of the whole Lie algebra; in this paper this function will be called *structure function.*

The eigenvector $|0\rangle$, corresponding to the zero eigenvalue of the number operator N, satisfies the following relation:

$$\text{if } F(0) = 0 \text{ (or } f(0) = 0) \text{ then } a|0\rangle = 0. \tag{12}$$

The structure function can be normalized such that $F(1) = 1$.

The eigenvectors of the number operator $N=f(a^\dagger a)$ are generated by the formula

$$|n\rangle=\frac{1}{\sqrt{[n]!}}(a^\dagger)^n|0\rangle \tag{13}$$

where

$$[n]!=\prod_{k=1}^{n}[k]=\prod_{k=1}^{n}F(k).$$

These eigenvectors are also eigenvectors of the *energy* operator

$$H=\frac{A}{2}(a^\dagger a+aa^\dagger) \tag{14}$$

corresponding to the eigenvalues

$$E_n=\frac{A}{2}([n+1]+[n])=\frac{A}{2}(F(n+1)+F(n)). \tag{15}$$

Consider that a given energy spectrum is defined by a real function $H(x)$ such that

$$E_n=\frac{A}{2}H(n+\tfrac{1}{2})$$

then

$$H(x+\tfrac{1}{2})=(F(x+1)+F(x)). \tag{16}$$

The general solution of this functional equation is a quite complicated task. Special cases of this equation were studied by Buck (1946) as quoted by Boas and Buck (1964).

3. The deformed oscillator equivalent to the Coulomb spectrum

Consider the case of a shifted Coulomb potential with centrifugal term

$$V(r)=-\frac{Ze^2}{r}+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}-2\delta R\hbar Z^2 \tag{17}$$

where R is the associated Rydberg's constant

$$R=\frac{e^4m}{2\hbar^3}.$$

The energy spectrum is given by

$$E_n=-R\hbar Z^2\left(\frac{1}{(n+l+1)^2}+2\delta\right).$$

Here we shall fix

$$\delta=\beta'(l+1) \tag{18}$$

for reasons which will be explained below. The function $\beta'(x)$ is the derivative of the beta function $\beta(x)$ defined as

$$\beta(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(x+k)} \qquad \text{and} \qquad \beta'(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(x+k)^2}$$

(see Gradshteyn and Ryzhik 1980, formula 8.372.1). After putting

$$\gamma = -\beta'(l+2) + \beta'(l+1)$$

we find

$$A = -2\gamma R\hbar Z^2 \qquad H(x) = \frac{1}{\gamma}\left(\frac{1}{(x+l+1/2)^2} + 2\delta\right)$$

while the function $F(x)$, which is the solution of equation (16), exists and is given by

$$F(x) = \frac{1}{\gamma}(-\beta'(x+l+1)+\delta). \tag{19}$$

In order to have $F(0)=0$, the constant δ in the above equation, should be given by the relation (18), then the shift of the Coulomb potential (17) is explained. Also the value of the constant γ is determined by the fact that the structure function $F(x)$ is normalized to unity, as $F(1)=1$. The structure function is an increasing continuous function for $x \geqslant 0$ and for $x \to \infty$ converges to $[\infty]$ where

$$[\infty] \equiv \lim_{x\to\infty} F(x) = \delta/\gamma.$$

The structure function determines uniquely the matrix representation of the Fock space $\{|n\rangle\}$, $n = 0, 1, \ldots, \infty$, then a matrix representation of the operators a and $a^\dagger$, N and H can be constructed:

$$\begin{aligned}
\langle n|a^\dagger|m\rangle &= \sqrt{[m+1]}\,\delta_{n,m+1}\\
\langle n|H|m\rangle &= -R\hbar Z^2\left(\frac{1}{(m+l+1)^2} + 2\delta\right)\delta_{n,m}\\
\langle n|N|m\rangle &= n\delta_{n,m}.
\end{aligned} \tag{20}$$

Therefore a destruction–creation operator algebra is fixed; this algebra is the analogous algebra of the Coulomb potential as it happens in the case of the harmonic oscillator potential.

4. Polynomial basis for the generalized deformed oscillator

In this section we shall discuss the polynomial basis of the deformed oscillator equivalent to the Coulomb potential. Let $\mathcal{H}$ be the set of the analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defined inside the circle $|z| < [\infty]$. Let the projection operator J_k project the function $f(z)$ to the truncated polynomial $J_k f(z)$ of degree k

$$J_k f(z) = \sum_{n=0}^{k} a_n z^n \in J_k\mathcal{H}.$$

The space spanned by the deformed oscillator basis $|n\rangle$ is equivalent to the space $\mathcal{H}$ spanned by the basis

$$\frac{z^n}{\sqrt{[n]!}} \qquad n = 0, 1, \ldots, \infty.$$

Any function $f(z)\in\mathscr{H}$ can be written as

$$f(z)=\sum_{n=0}^{\infty} f_n \frac{z^n}{[n]!}\equiv\langle z|f\rangle \tag{21}$$

where

$$\langle z|\equiv\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}\langle n| \tag{22}$$

and

$$|f\rangle\equiv\sum_{n=0}^{\infty}\frac{f_n}{\sqrt{[n]!}}|n\rangle. \tag{23}$$

The series (21) converge for $|z|<[\infty]$ if

$$\lim_{n\to\infty}\frac{|f_{n+1}|}{|f_n|}<\infty.$$

The element $|z\rangle$ is the *coherent* (but not normalized) eigenstate of the destruction operator a with eigenvalue $\bar{z}$:

$$a|z\rangle=\bar{z}|z\rangle. \tag{24}$$

The multiplication of the function $f(z)$ by z can be regarded as an application from the space of functions $\mathscr{H}$ into $\mathscr{H}$, and this operation corresponds to the following one:

$$z\sum_{n=0}^{\infty} a_n z^n=\sum_{n=1}^{\infty} a_{n-1}z^n\in\mathscr{H}-J_0\mathscr{H}. \tag{25}$$

The derivative $\partial/\partial z$ is also an application defined in the space $\mathscr{H}$:

$$(\partial/\partial z)\sum_{n=0}^{\infty} a_n z^n=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n\in\mathscr{H}. \tag{26}$$

In the space $\mathscr{H}$ we can define the operator

$$\frac{\partial}{\partial_D z}\equiv\frac{1}{z}F\left(z\circ\frac{\partial}{\partial z}\right). \tag{27}$$

Without difficulty we can show that

$$\frac{\partial}{\partial_D z}z^n=F(n)z^{n-1}\qquad\qquad \frac{\partial}{\partial_D z}z^0=0.$$

The deformed derivative $(\partial/\partial_D z)$ can be constructed analytically. Consider the integration operator Int acting on $\mathscr{H}$ as follows:

$$\operatorname{Int} f(z)\equiv\int_0^z f(u)\,\mathrm{d}u.$$

From the definition (19) of the structure function $F(x)$ the deformed derivative is shown to be

$$\frac{\partial}{\partial_D z}\equiv\frac{1}{z\gamma}\left[\sum_{k=0}^{\infty}(-1)^k(z^{-(l+k+1)}\circ\operatorname{Int}\circ z^{l+k})^2+\delta\right]. \tag{28}$$

The expansion (21) of the function $f(z)$ corresponds to the deformed Taylor expansion around zero because we can easily show that

$$f_n = \left[\left(\frac{\partial}{\partial_D u}\right)^n f(u)\right]_{u=0} \equiv f^{[n]}(0). \tag{29}$$

The operator $z \circ \partial/\partial_D z$ is a one-to-one application in the subspace $\mathscr{H} - J_0\mathscr{H}$, therefore the inverse of this operator exists in this subspace and is given by

$$(z \circ \partial/\partial_D z)^{-1} z^n \equiv \frac{1}{F(n)} z^n.$$

Using the above operator the integration operator Int_D can be defined by

$$\mathrm{Int}_D \equiv (z \circ \partial/\partial_D z)^{-1} \circ z. \tag{30}$$

Without difficulty the following relation can be shown:

$$\mathrm{Int}_D\, z^n = \frac{z^{n+1}}{F(n+1)}.$$

For any function $f(z)$, given by equation (21), we can define the integral

$$\mathrm{Int}_D f(z) \equiv \int_0^z f(u)\, \mathrm{d}_D u = \sum_{n=1}^{\infty} f_{n-1} \frac{z^n}{[n]!}$$

and by definition

$$\int_a^b f(u)\, \mathrm{d}_D u = \mathrm{Int}_D f(b) - \mathrm{Int}_D f(a). \tag{31}$$

The definition of the coherent state (22) and (24) implies that the deformed exponential function is defined by the following formula:

$$\exp_D(z\bar{w}) \equiv \langle z|w\rangle = \sum_{n=0}^{\infty} \frac{(z\bar{w})^n}{[n]!}. \tag{32}$$

The function $\exp_D(z)$ converges if $|z| < [\infty]$. Without difficulty we can show the deformed generalizations of the usual identities:

$$\int_0^z \left(\frac{\partial}{\partial_D u}\right) f(u)\, \mathrm{d}_D u = f(z) - f(0)$$

$$\left(\frac{\partial}{\partial_D z}\right) \int_0^z f(u)\, \mathrm{d}_D u = f(z)$$

$$\left(\frac{\partial}{\partial_D w}\right)^n \exp_D[wz] = z^n \exp_D[wz]$$

$$\int_0^z \exp_D[wu]\, \mathrm{d}_D u = w^{-1}(\exp_D[wz] - 1)$$

$$\int_0^z u^n \exp_D[wu]\, \mathrm{d}_D u = \left(\frac{\partial}{\partial_D w}\right)^n [w^{-1}(\exp_D[wz] - 1)].$$

The space $\mathscr{H}$ has also the structure of a Hilbert space with the product

$$\langle f|g\rangle \equiv \left[\bar{f}\left(\frac{\partial}{\partial_D z}\right) g(z)\right]_{z=0}. \tag{33}$$

This structure is compatible with the expressions (21)–(23).

The above defined product can be formally generated by introducing a measure $d\mu(\bar{z}, z)$ on the complex z plane having the following properties:

$$\int d\mu(\bar{z}, z)\bar{z}^n z^m \equiv \delta_{n,m}[n]! \tag{34}$$

and

$$\langle f|g\rangle = \int d\mu(\bar{z}, z)\bar{f}(\bar{z})g(z). \tag{35}$$

The basic properties of this measure can be easily deduced from equation (34); we shall list here only the most fundamental ones. if A is an operator defined on the finite dimensional Hilbert space spanned by the vectors $|n\rangle$ then there is a matrix representation defined by

$$A = \sum_{m,n=0}^{\infty} A_{n,m}|n\rangle\langle m|. \tag{36}$$

This operator corresponds to a kernel $A(w, u)$ acting on the space $\mathscr{H}$

$$\begin{aligned} A(w, \bar{u}) \equiv \langle w|A|u\rangle &= \sum_{m,n=0}^{\infty} A_{n,m}\langle w|n\rangle\langle m|u\rangle \\ &= \sum_{m,n=0}^{\infty} A_{n,m}\frac{w^n}{\sqrt{[n]!}}\frac{\bar{u}^m}{\sqrt{[m]!}} \end{aligned} \tag{37}$$

and

$$\langle w|A|f\rangle = \int d\mu(\bar{z}, z)A(w, \bar{z})f(z) \tag{38}$$

where the function $f(z)$ is defined by equations (21) and (23).

The product of two operators A and B corresponds to the *convolution* of the corresponding kernels

$$\langle w|AB|u\rangle = \int d\mu(\bar{z}, z)A(w, \bar{z})B(z, \bar{u}). \tag{39}$$

This equation implies the following *resolution* of the identity

$$\int d\mu(\bar{z}, z)|z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| \equiv \mathbf{1} \tag{40}$$

where $\mathbf{1}$ is the unity in the Hilbert space spanned by the vector basis $|n\rangle$. The following relations can be proved without difficulty, using the definition (34) of the deformed exponential function:

$$\int d\mu(\bar{z}, z)\exp_D(u\bar{z})f(z) = f(u)$$

$$\int d\mu(\bar{z}, z)\exp_D(u\bar{z})\exp_D(z\bar{w}) = \exp_D(u\bar{w})$$

$$\int d\mu(\bar{z}, z)\exp_D(z\bar{z}t) = \sum_{n=0}^{\infty} t^n = (1-t)^{-1}.$$

By analogy, as in the q-analysis case (see Exton 1983), we can define the *binomial* function

$$\Phi(\alpha;t)=\sum_{n=0}^{\infty}\frac{[\alpha][\alpha+1]\ldots[\alpha+n-1]}{[n]!}t^n.$$

If $[x]=F(x)=1$ the above sum is equal to $(1-t)^{-\alpha}$, while if $[x]=F(x)=(q^x-1)/(q-1)$ the corresponding formula for the q-analysis case is found (see Exton 1983, p. 120). The binomial case is a special case of the deformed hypergeometric function, but this subject will not be studied here. Using the binomial function after lengthy calculations, one can find the generalization of the Gauss integral

$$\int \mathrm{d}\mu(\bar{z},z)\exp_D(\bar{b}z)\exp_D(z\bar{z}t)\exp_D(\bar{z}a)=\sum_{n=0}^{\infty}\frac{(a\bar{b})^n}{[n]!}\Phi(1+n;t).$$

In the case of the usual harmonic oscillator $(F(x)=x)$, the right-hand side of this formula is reduced to $\exp(a\bar{b}/(1-t))/(1-t)$, in this case $\mathrm{d}\mu(\bar{z},z)=\exp(-|z|^2)\,\mathrm{d}\bar{z}\,\mathrm{d}z/\pi$. If A is an operator defined by equations (36), (37), then

$$\int \mathrm{d}\mu(\bar{z},z)A(z,\bar{z})=\mathrm{Tr}(A)=\sum_{n=0}^{\infty}A_{n,n}.$$

These formulae indicate that the measure $\mathrm{d}\mu(\bar{z},z)$ has the basic properties of a Gaussian measure.

The interesting problem, which consistently arises, is to clarify the connection between the Hilbert space spanned by the eigenvectors $|n\rangle$ and the eigenfunctions of the Schrödinger equation with a Coulomb potential. The corresponding problem for the ordinary harmonic oscillator is the well known correspondence between the eigenstates $|n\rangle$ in the creation–destruction formalism and the Hermite polynomials weighted by a Gaussian measure, which are the eigenfunctions of the Schrödinger equation.

All the formulae of this section (except equation (28)) are independent of the choice of the structure function $F(x)$; these formulae are general and applicable for a spectrum with denombrable but infinite cardinal number (Coulomb, harmonic oscillator potential or q-deformed oscillator, with q being a real number). In the case of a finite spectrum (Pöschl–Teller potential or q-deformed oscillator with q being a root of unity) the corresponding formulae are given by Daskaloyannis (1992).

5. Concluding remarks

In this paper we have constructed an algebra of operators

$$\{a,a^{\dagger},N,1\}$$

satisfying the anticommutation relations

$$[a,N]=a \qquad \text{and} \qquad [a^{\dagger},N]=-a^{\dagger}$$

$$[a,a^{\dagger}]=F(N+1)-F(N).$$

The structure function $F(x)$ is given by equation (19), then this algebra corresponds to the energy spectrum of the Coulomb potential. The polynomial basis and the associated deformed integration and derivation are constructed. The formulae given in section 4 are quite general and they can be applied to any case of energy spectra with an infinite number of energy eigenvalues with an accumulation point finite or infinite.

We must point out that there is a deformed analysis appropriate to the Coulomb deformed oscillator, as the q-deformed analysis corresponds to the q-deformed oscillator algebra or the usual analysis is appropriate to the ordinary oscillator algebra. In this kind of analysis there is not defined a simple Leibnitz rule of the differentiation, although some of the results of the usual analysis can be produced for different kinds of deformed analysis. This topic seems to be an interesting topic for a forthcoming publication.

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